

COMPOUND BASIS ARISING FROM THE BASIC $A_1^{(1)}$ -MODULE

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ABSTRACT. A new basis for the polynomial ring of infinitely many variables is constructed which consists of products of Schur functions and Q -functions. The transition matrix from the natural Schur function basis is investigated.

1. INTRODUCTION

This note concerns with realizations of the basic representation of the affine Lie algebra of type $A_1^{(1)}$ (cf. [6]). The most well-known realization is PU , principal, untwisted, whose representation space is

$$\mathcal{F}^{PU} = \mathbb{C}[t_j; j \geq 1, \text{odd}].$$

In the context of nonlinear integrable systems, this space appears as that of the KdV hierarchy. The second one is HU , homogeneous, untwisted, which is on

$$\mathcal{F}^{HU} = \bigoplus_{m \in \mathbb{Z}} \mathcal{F}(m); \quad \mathcal{F}(m) = \mathbb{C}[t_j; j \geq 1] \otimes q^m.$$

This space is for the NLS (nonlinear Schrödinger) hierarchy and also for the Fock representation of the Virasoro algebra (cf. [5]). The third one is PT , principal, twisted, on \mathcal{F}^{PT} which coincides with \mathcal{F}^{PU} . And the fourth one is HT , homogeneous, twisted, on \mathcal{F}^{HT} which is the same as \mathcal{F}^{HU} . The Lie algebra of type $A_1^{(1)}$ is isomorphic to that of type $D_2^{(2)}$. One can discuss twisted realization of $A_1^{(1)}$ -modules via this isomorphism.

The purpose of this note is to give a weight basis for \mathcal{F}^{HT} and compare it with a standard Schur function basis for \mathcal{F}^{HU} . We will show that the transition matrix has several interesting combinatorial features. This is a detailed version of our announcement [1].

2. A QUICK REVIEW OF REALIZATIONS

Let us first consider the principal untwisted realization on $\mathcal{F}^{PU} = \mathbb{C}[t_j; j \geq 1, \text{odd}]$. To describe a weight basis for this space we need Schur functions and Schur's Q -functions in our setting. Let P_n be the set of all partitions of n and put $P = \bigcup_{n \geq 0} P_n$. For $\lambda \in P_n$, the Schur function $S_\lambda(t)$ is defined by

$$S_\lambda(t) = \sum_{\rho=(1^{m_1} 2^{m_2} \dots) \in P_n} \chi_\rho^\lambda \frac{t_1^{m_1} t_2^{m_2} \dots}{m_1! m_2! \dots},$$

where the summation runs over all partitions $\rho = (1^{m_1} 2^{m_2} \dots)$ of n , and χ_ρ^λ is the irreducible character of the symmetric group \mathfrak{S}_n , indexed by λ and evaluated at the conjugacy class ρ . The Schur functions are the ordinary irreducible characters of the general linear groups. If the group element g has eigenvalues x_1, x_2, \dots , then

the original irreducible character is recovered by putting $p_j := jt_j$ ($j \geq 1$), where $p_j = \sum_{i \geq 1} x_i^j$ is the j -th power sum of the eigenvalues.

The 2-reduction of a polynomial $f(t)$ is to “kill” the even numbered variables t_2, t_4, \dots , i.e. ,

$$f^{(2)}(t) = f(t)|_{t_2=t_4=\dots=0} \in \mathcal{F}^{PU}.$$

The 2-reduced Schur functions are linearly dependent in general. However all linear relations among them are known, and one can choose certain set $P' \subset P$ so that $\{S_\lambda^{(2)}; \lambda \in P'\}$ forms a basis for \mathcal{F}^{PU} (cf. [2]).

The space \mathcal{F}^{PU} also affords the principal twisted realization. A weight basis is best described by Schur’s Q -functions. Let SP_n (resp. OP_n) be the set of all strict (resp. odd) partitions of n and put $SP = \bigcup_{n \geq 0} SP_n$, $OP = \bigcup_{n \geq 0} OP_n$. For $\lambda \in SP_n$, the Q -function $Q_\lambda(t)$ is defined by

$$Q_\lambda(t) = \sum_{\rho=(1^{m_1}3^{m_3}\dots) \in OP_n} 2^{\frac{\ell(\lambda)-\ell(\rho)+\epsilon}{2}} \zeta_\rho^\lambda \frac{t_1^{m_1} t_3^{m_3} \dots}{m_1! m_3! \dots},$$

where the summation runs over all odd partitions $\rho = (1^{m_1}3^{m_3}\dots)$ of n , $\epsilon = 0$ or 1 according to that $\ell(\lambda) - \ell(\rho)$ is even or odd and ζ_ρ^λ is the irreducible spin character of \mathfrak{S}_n , indexed by λ and evaluated at the conjugacy class ρ . For the Q -functions, we set $p_j := \frac{1}{2}jt_j$ ($j \geq 1$, odd) as the relation with the “eigenvalues”. A more detailed account is found in [9]. Here we remark the relation of Q -functions and the P -functions. We define inner product $\langle \cdot, \cdot \rangle_q$ on $\mathcal{F}(0)$ by $\langle p_\lambda, p_\mu \rangle_q = z_\lambda(q) \delta_{\lambda\mu}$, where $z_\lambda(q) = z_\lambda \prod_{i \geq 1} (1 - q^{\lambda_i})^{-1}$. Note that $z_\lambda(-1)$ cannot be defined for λ which has even parts. Therefore we have to re-define $\langle \cdot, \cdot \rangle_{-1}$ by setting $\langle p_\lambda, p_\mu \rangle_{-1} = 2^{-\ell(\lambda)} z_\lambda \delta_{\lambda\mu}$. The P -functions are dual to the Q -functions with respect to the inner product $\langle \cdot, \cdot \rangle_{-1}$ on \mathcal{F}^{PU} . For a strict partition λ , we see that $P_\lambda(t) = 2^{-\ell(\lambda)} Q_\lambda(t)$ (cf. [8]).

In order to give the homogeneous, twisted realization we employ a combinatorics of strict partitions. We introduce the following h-abacus. For example, the h-abacus of $\lambda = (11, 10, 5, 3, 2)$ is shown below.

$$\begin{array}{ccccc} & & 1 & \textcircled{3} & \\ & & & & \\ \textcircled{2} & & & & \\ 4 & \textcircled{5} & & 7 & \\ 6 & & & & \\ 8 & 9 & \textcircled{11} & & \\ \textcircled{10} & & & & \\ 12 & 13 & 15 & & \\ \vdots & \vdots & \vdots & & \end{array}$$

From this h-abacus of λ we read off a triplet $(\lambda^{hc}; \lambda^h[0], \lambda^h[1])$ of partitions. Firstly $\lambda^h[0] = (5, 1)$, a strict partition obtained just by taking halves of the circled positions of the leftmost column.

For obtaining $\lambda^h[1]$, we need the following process:

- (1) For the third column, the circled positions correspond to the vacancies ”○”.
- (2) For the second column, the circled positions correspond to being occupied ”●”.

- (3) Read the third column from infinity to the position 3 and consequently the second column from the position 1 to infinity, and draw the Maya diagram

$$\begin{array}{cccccccc} \dots & 15 & 11 & 7 & 3 & 1 & 5 & 9 & \dots \\ & \bullet & \circ & \bullet & \circ & \circ & \bullet & \underline{\circ} & \end{array}$$

- (4) For each \bullet , count the number of vacancies which are on the left of that \bullet , and get a partition

$$\lambda^h[1] = (3, 1).$$

Next the h -core λ^{hc} is obtained by the following moving and removing:

- (1) Remove all circles on the leftmost column.
- (2) Move a circle one position up along the second or the third column.
- (3) Remove the two circles at the positions 1 and 3 simultaneously.
- (4) The “stalemate” determines the partition

$$\lambda^{hc} = (3).$$

Note that λ^{hc} is always of the form

$$\Delta^h(m) = (4m - 3, 4m - 7, \dots, 5, 1) \text{ or } \Delta^h(-m) = (4m - 1, 4m - 5, \dots, 7, 3)$$

for some $m \in \mathbb{N}$ ($\Delta^h(0) = \emptyset$). Let HC be the set of all such λ^{hc} 's. In this way we have a one-to-one correspondence between $\lambda \in SP$ and $(\lambda^{hc}; \lambda^h[0], \lambda^h[1]) \in HC \times SP \times P$ with the condition

$$|\lambda| = |\lambda^{hc}| + 2(|\lambda^h[0]| + 2|\lambda^h[1]|).$$

By making use of this one-to-one correspondence, we define the linear map $\eta : \mathcal{F}^{PT} \rightarrow \mathcal{F}^{HT}$ by

$$\eta(Q_\lambda(t)) = Q_{\lambda^h[0]}(t) S_{\lambda^h[1]}(t') \otimes q^{m(\lambda)}.$$

Here

$$m(\lambda) = (\text{number of circles on the second column}) - (\text{number of circles on the third column})$$

and $S_\nu(t') = S_\nu(t)|_{t_j \mapsto t_{2j}}$ for any $j \geq 1$. For any integer m , the set

$$\{\eta(Q_\lambda); \lambda \in SP, m(\lambda) = m\}$$

forms a basis for $\mathcal{F}(m) = \mathbb{C}[t_j; j \geq 1] \otimes q^m$ (cf. [4]). Under the condition $m = 0$, there is a one-to-one correspondence between the following two sets for any $n \geq 0$:

- (i) $\{\lambda \in SP_{2n}; \lambda^{hc} = \emptyset\}$,
- (ii) $\{(\mu, \nu) \in SP_{n_0} \times P_{n_1}; n_0 + 2n_1 = n\}$.

3. COMPOUND BASIS

We begin with some bijections between sets of partitions. The first one is

$$\phi : P_n \longrightarrow \bigcup_{n_0 + 2n_1 = n} SP_{n_0} \times P_{n_1}$$

defined by $\lambda \mapsto (\lambda^r, \lambda^d)$. Here the multiplicities $m_i(\lambda^r)$ and $m_i(\lambda^d)$ of $i \geq 1$ are given respectively by

$$m_i(\lambda^r) = \begin{cases} 1 & m_i(\lambda) \equiv 1 \pmod{2} \\ 0 & m_i(\lambda) \equiv 0 \pmod{2}, \end{cases}$$

and

$$m_i(\lambda^d) = \begin{cases} \frac{1}{2}(m_i(\lambda) - 1) & m_i(\lambda) \equiv 1 \pmod{2} \\ \frac{1}{2}m_i(\lambda) & m_i(\lambda) \equiv 0 \pmod{2}. \end{cases}$$

For example, if $\lambda = (5^3 4^4 2^7 1)$, then $\lambda^r = (521)$ and $\lambda^d = (54^2 2^3)$. We set

$$P_{n_0, n_1} = \phi^{-1}(SP_{n_0} \times P_{n_1}).$$

The second bijection is

$$\psi : P_n \longrightarrow \bigcup_{n_1+2n_2=n} OP_{n_1} \times P_{n_2}$$

defined by $\psi(\lambda) = (\lambda^o, \lambda^e)$. Here λ^o is obtained by picking up the odd parts of λ , while λ^e is obtained by taking halves of the even parts. For example, if $\lambda = (5^3 4^4 2^7 1)$, then $\lambda^o = (5^3 1)$ and $\lambda^e = (2^4 1^7)$.

The third bijection is called the Glaisher map. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a strict partition of n . Suppose that $\lambda_i = 2^{p_i} q_i$ ($i = 1, 2, \dots$), where q_i is odd. Then an odd partition $\tilde{\lambda}$ of n is defined by

$$m_{2j-1}(\tilde{\lambda}) = \sum_{q_i=2j-1, i \geq 1} 2^{p_i}.$$

For example, if $\lambda = (8, 6, 4, 3, 1)$, then $\tilde{\lambda} = (3^3, 1^{13})$. This gives a bijection between SP_n and OP_n .

Proposition 3.1. *Let (n_0, n_1) be fixed. Then we have*

$$\begin{aligned} \sum_{\lambda \in P_n} \ell(\lambda) &= \sum_{\lambda \in P_n} (\ell(\lambda^r) + 2\ell(\lambda^d)) = \sum_{\lambda \in P_n} (\ell(\lambda^o) + \ell(\lambda^e)) = \sum_{\lambda \in P_n} (\ell(\tilde{\lambda}^r) + \ell(\lambda^e)), \\ \sum_{\lambda \in P_{n_0, n_1}} \ell(\lambda) &= \sum_{\lambda \in P_{n_0, n_1}} (\ell(\lambda^r) + 2\ell(\lambda^d)) = \sum_{\lambda \in P_{n_0, n_1}} (\ell(\lambda^o) + \ell(\lambda^e)), \\ \sum_{\lambda \in P_n} 2\ell(\lambda^d) &= \sum_{\lambda \in P_n} 2\ell(\lambda^e) = \sum_{\lambda \in P_n} (\ell(\lambda^o) + \ell(\lambda^e) - \ell(\lambda^r)) = \sum_{\lambda \in P_n} (\ell(\tilde{\lambda}^r) + \ell(\lambda^e) - \ell(\lambda^r)), \\ \text{and} \\ \sum_{\lambda \in P_{n_0, n_1}} 2\ell(\lambda^d) &= \sum_{\lambda \in P_{n_0, n_1}} (\ell(\lambda^o) + \ell(\lambda^e) - \ell(\lambda^r)). \end{aligned}$$

Looking at the representation spaces \mathcal{F}^{HU} and \mathcal{F}^{HT} , we have the following two natural bases for the space

$$\mathcal{F}(0)_n = \mathbb{C}[t_j; j \geq 1]_n$$

consisting of the homogenous polynomials of degree n subject to $\deg t_j = j$. Namely we have

- (i) $\{S_\lambda(t); \lambda \in P_n\}$,
- (ii) $\{Q_{\lambda^r}(t)S_{\lambda^d}(t'); \lambda \in P_n\}$.

For simplicity we write

$$W_\lambda(t) = Q_{\lambda^r}(t)S_{\lambda^d}(t')$$

for $\lambda \in P_n$ and call the set (ii) the compound basis for $\mathcal{F}(0)_n$.

Our problem is to determine the transition matrix between these two bases. Let $A_n = (a_{\lambda\mu})$ be defined by

$$(1) \quad S_\lambda(t) = \sum_{\mu \in P_n} a_{\lambda\mu} W_\mu(t)$$

for $\lambda \in P_n$.

Here we remark the relation between our basis and the Q' -functions. Lascoux, Leclerc and Thibon (cf. [7]) introduced the Q' -functions as the basis for $\mathcal{F}(0)_n$ dual to P -functions with respect to the inner product

$$\langle F(t), G(t) \rangle_0 := F(\tilde{\partial}) \overline{G(t)}|_{t=0},$$

where $\tilde{\partial} = (\frac{\partial}{\partial t_1}, \frac{1}{2} \frac{\partial}{\partial t_2}, \frac{1}{3} \frac{\partial}{\partial t_3}, \dots)$. For a strict partition μ we see that $Q'_\mu(t) = Q_\mu(2t)$. For a partition λ which is not necessarily strict, we see that

$$Q'_\lambda(t) = Q_{\lambda^r}(2t) h_{\lambda^d}(t')$$

where h_{λ^d} is the complete symmetric function indexed by λ^d . Therefore the transition from W_λ to Q'_μ is essentially given by the Kostka numbers.

4. TRANSITION MATRICES

In the previous section, functions are expressed in terms of the “time variables” $t = (t_1, t_2, \dots)$ of the soliton equations. However, for the description and the proof of our formula, it is more convenient to use the “original” variables of the symmetric functions, i.e., the eigenvalues $x = (x_1, x_2, \dots)$.

The definition (1) of $a_{\lambda\mu}$ is rewritten as

$$S_\lambda(x, x) = \sum_{\mu \in P_n} a_{\lambda\mu} Q_{\mu^r}(x) S_{\mu^d}(x^2),$$

where $(x, x) = (x_1, x_1, x_2, x_2, \dots)$ and $x^2 = (x_1^2, x_2^2, \dots)$. Hereafter we will denote

$$W_\lambda(x) = Q_{\lambda^r}(x) S_{\lambda^d}(x^2), \quad V_\lambda(x) = P_{\lambda^r}(x) S_{\lambda^d}(x^2).$$

Also we set the following spaces of symmetric functions

$$\Lambda = \mathbb{C}[p_r(x); r \geq 1], \quad \Gamma = \mathbb{C}[p_r(x); r \geq 1, \text{ odd}],$$

and

$$\Gamma' = \mathbb{C}[p_r(x); r \geq 2, \text{ even}]$$

so that

$$\Lambda \cong \Gamma \otimes \Gamma'.$$

We have two bases for Λ :

$$W = (W_\lambda(x))_\lambda \text{ and } V = (V_\lambda(x))_\lambda.$$

First we notice the following Cauchy identity.

Proposition 4.1.

$$\prod_{i,j \geq 1} \frac{1}{(1 - x_i y_j)^2} = \sum_{\lambda \in P} W_\lambda(x) V_\lambda(y).$$

Proof. We compute

$$\begin{aligned} \sum_{\lambda \in P} W_\lambda(x) V_\lambda(y) &= \sum_{\lambda \in P} Q_{\lambda^r}(x) S_{\lambda^d}(x^2) P_{\lambda^r}(y) S_{\lambda^d}(y^2) \\ &= \sum_{\mu \in SP} Q_\mu(x) P_\mu(y) \sum_{\nu \in P} S_\nu(x^2) S_\nu(y^2). \end{aligned}$$

Taking the inner products $\langle \cdot, \cdot \rangle_{-1}$ and $\langle \cdot, \cdot \rangle_0$ on Λ , we obtain

$$\sum_{\mu \in SP} Q_\mu(x) P_\mu(y) = \prod_{i,j} \frac{1 + x_i y_j}{1 - x_i y_j},$$

and

$$\sum_{\nu \in P} S_\nu(x^2) S_\nu(y^2) = \prod_{i,j} \frac{1}{1 - x_i^2 y_j^2}.$$

We have

$$\sum_{\lambda \in P} W_\lambda(x) V_\lambda(y) = \prod_{i,j} \frac{1}{(1 - x_i y_j)^2}.$$

□

By a standard argument, we have

Corollary 4.2.

$$\langle W_\lambda(x), V_\mu(x) \rangle_{-1} = \delta_{\lambda\mu}.$$

Theorem 4.3. *The matrix A_n is integral.*

Proof. We have

$$\sum_{\lambda \in P} W_\lambda(x) V_\lambda(y) = \prod_{i,j} \frac{1}{(1 - x_i y_j)^2} = \sum_{\lambda \in P} S_\lambda(x, x) S_\lambda(y).$$

Taking the inner product $\langle \cdot, \cdot \rangle_0$ with $S_\mu(y)$, we obtain

$$\begin{aligned} S_\lambda(x, x) &= \sum_{\mu \in P} \langle W_\mu(x) V_\mu(y), S_\lambda(y) \rangle_0 \\ &= \sum_{\mu \in P} \langle V_\mu(y), S_\lambda(y) \rangle_0 W_\mu(x). \end{aligned}$$

Thus we know

$$a_{\lambda\mu} = \langle V_\mu(y), S_\lambda(y) \rangle_0.$$

The numbers $g_{\mu^r \nu}$ defined by

$$P_{\mu^r}(y) = \sum_{\nu \in P} g_{\mu^r \nu} S_\nu(y)$$

are called the Stembridge coefficients and are known to be non-negative integers. Also one finds the following formula in [3].

$$S_{\mu^p}(y^2) = \sum_{\xi \in P} \delta(\xi) c_{\xi[0], \xi[1]}^{\mu^d} S_\xi(y),$$

where $\delta(\xi)$ is the 2-*sign* of ξ , $(\xi[0], \xi[1])$ is the 2-*quotient* of ξ (cf. [10]) and $c_{\xi[0], \xi[1]}^{\mu^d}$ is the Littlewood-Richardson coefficient. Hence

$$\begin{aligned} V_\mu(y) &= P_{\mu^r}(y) S_{\mu^p}(y^2) \\ &= \sum_{\nu, \xi} \delta(\xi) g_{\mu^r \nu} c_{\xi[0], \xi[1]}^{\mu^d} S_\nu(y) S_\xi(y) \\ &= \sum_{\lambda} \left(\sum_{\nu, \xi} \delta(\xi) g_{\mu^r \nu} c_{\nu \xi}^{\lambda} c_{\xi[0], \xi[1]}^{\mu^d} \right) S_\lambda(y). \end{aligned}$$

Therefore

$$a_{\lambda \mu} = \sum_{\nu, \xi} \delta(\xi) g_{\mu^r \nu} c_{\nu \xi}^{\lambda} c_{\xi[0], \xi[1]}^{\mu^d}$$

is an integer. □

Example 4.4.

$$\begin{aligned} A_3 &= \begin{matrix} & (3, \emptyset) & (21, \emptyset) & (1, 1) \\ \begin{pmatrix} (3) \\ (21) \\ (1^3) \end{pmatrix} & \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \end{matrix} \\ A_4 &= \begin{matrix} & (4, \emptyset) & (31, \emptyset) & (\emptyset, 2) & (\emptyset, 1^2) & (2, 1) \\ \begin{pmatrix} (4) \\ (31) \\ (2^2) \\ (1^4) \\ (21^2) \end{pmatrix} & \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 & -1 \end{pmatrix} \end{matrix} \end{aligned}$$

As for the columns corresponding to (μ, \emptyset) with $\mu \in SP_n$, entries are non-negative integers. The submatrix consisting of these columns will be denoted by Γ_n . The entries of Γ_n are the Stembridge coefficients, whose combinatorial nature has been known ([11], [8]).

Here we recall the definition of decomposition matrices for the p -modular representations of the symmetric group S_n . Let p be a fixed prime number. A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is said to be p -regular if there are no parts satisfying $\lambda_i = \lambda_{i+1} = \dots = \lambda_{i+p-1} \geq 1$. Note that a 2-regular partition is nothing but a strict partition. The set of p -regular partitions of n is denoted by $P_n^{r(p)}$. A partition $\rho = (1^{m_1} 2^{m_2} \dots)$ is said to be p -class regular if $m_p = m_{2p} = \dots = 0$. Note that a 2-class regular partition is nothing but an odd partition. The set of p -class regular partitions of n is denoted by $P_n^{c(p)}$. The p -Glaisher map $\lambda \mapsto \tilde{\lambda}$ is defined in a natural way. This gives a bijection between $P_n^{r(p)}$ and $P_n^{c(p)}$. For $\lambda \in P_n^{r(p)}$, we define the Brauer-Schur function $B_\lambda^{(p)}(t)$ indexed by λ as follows.

$$B_\lambda^{(p)}(t) = \sum_{\rho \in P_n^{c(p)}} \varphi_\rho^\lambda \frac{t_1^{m_1} t_2^{m_2} \dots}{m_1! m_2! \dots} \in \mathcal{F}(0)_n,$$

where φ_ρ^λ is the irreducible Brauer character corresponding to λ , evaluated at the p -regular conjugacy class ρ . These functions form a basis for the space $\mathcal{F}_n^{(p)} =$

$\mathcal{F}^{(p)} \cap \mathcal{F}(0)_n$, where

$$\mathcal{F}^{(p)} = \mathbb{C}[t_j; j \geq 1, j \not\equiv 0 \pmod{p}].$$

Given a Schur function $S_\lambda(t)$, define the p -reduced Schur function $S_\lambda^{(p)}(t)$ by "killing" all variables t_p, t_{2p}, \dots ;

$$S_\lambda^{(p)}(t) = S_\lambda(t)|_{t_{jp}=0}.$$

These p -reduced Schur functions are no longer linearly independent. All linear relations among these polynomials are known (cf. [2]). The p -decomposition matrix $D_n^{(p)} = (d_{\lambda\mu})$ is defined by

$$S_\lambda^{(p)}(t) = \sum_{\mu \in P_n^{(p)}} d_{\lambda\mu} B_\mu^{(p)}(t)$$

for $\lambda \in P_n$, and are known to satisfy the properties; $d_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$, $d_{\lambda\mu} = 0$ unless $\mu \geq \lambda$ and $d_{\lambda\lambda} = 1$. Here " \geq " denotes the dominance order.

Now let us go back to the case of $p = 2$. By definition, the Stembridge coefficients $\gamma_{\lambda\mu}$ ($\lambda \in P_n, \mu \in SP_n$) appear as

$$S_\lambda^{(2)}(t) = \sum_{\mu \in SP_n} \gamma_{\lambda\mu} Q_\mu(t).$$

Looking at the matrices $D_n^{(2)} = (d_{\lambda\mu})$ and $\Gamma_n = (\gamma_{\lambda\mu})$, one observes that they are "very similar". We consider the Cartan matrix $C_n^{(2)} = {}^t D_n^{(2)} D_n^{(2)}$ and the correspondent $G_n = {}^t \Gamma_n \Gamma_n$. There is a compact formula for the elementary divisors of $C_n^{(2)}$ ([12]): $2^{\ell(\tilde{\lambda}) - \ell(\lambda)}$ for $\lambda \in SP_n$.

Theorem 4.5. *The elementary divisors of $C_n^{(2)}$ and G_n coincide.*

Proof. We put $\tilde{Z}_n = (2^{\frac{\ell(\lambda) - \ell(\rho) + \epsilon}{2}} \zeta_\rho^\lambda)_{\lambda \in SP_n, \rho \in OP_n}$, $\Phi_n^{(2)} = (\varphi_\rho^\lambda)_{\lambda \in SP_n, \rho \in OP_n}$ and $X_n^{(2)} = (\chi_\rho^\lambda)_{\lambda \in P_n, \rho \in OP_n}$. The transition matrix $T_n = (t_{\lambda\mu})_{\lambda, \mu \in SP_n}$ is defined by

$$B_\lambda^{(2)}(t) = \sum_{\mu \in SP(n)} t_{\lambda\mu} Q_\mu(t).$$

By definition of Γ_n and $D_n^{(2)}$, we have $X_n^{(2)} = \Gamma_n \tilde{Z}_n = D_n^{(2)} \Phi_n^{(2)}$ and $\Phi_n^{(2)} = T_n \tilde{Z}_n$. Hence, we have $\Gamma_n = X_n^{(2)} \tilde{Z}_n^{-1} = D_n^{(2)} \Phi_n^{(2)} \tilde{Z}_n^{-1} = D_n^{(2)} T_n$.

The matrix Γ_n has the following properties; $\gamma_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$, $\gamma_{\lambda\mu} = 0$ unless $\mu \geq \lambda$ and $\gamma_{\lambda\lambda} = 1$ ([11]). Fix a total order in the set of partitions which is compatible with the dominance order, and we shall write (d_{ij}) , (γ_{ij}) and (t_{ij}) in place of $(d_{\lambda\mu})$, $(\gamma_{\lambda\mu})$ and $(t_{\lambda\mu})$, respectively. Looking at the first row of $D_n^{(2)} T_n$, we have

$$\delta_{1j} = \gamma_{1j} = \sum_{k=1} d_{1k} t_{kj}.$$

This shows that $t_{1j} = \delta_{1j}$. As for the second row of $D_n^{(2)} T_n$, we have

$$\delta_{2j} = \gamma_{2j} = \sum_{k=1} d_{2k} t_{kj} = \sum_{k=2} d_{2k} t_{kj} \quad (j \geq 2).$$

This shows that $t_{2j} = \delta_{2j}$. Inductively, we can see that T_n is a lower unitriangular integral matrix. Therefore the matrix $D_n^{(2)}$ and Γ_n are transformed to each other by

column operations. By a standard argument we see that the elementary divisors of $C_n^{(2)}$ and G_n coincide. \square

Our transition matrix $A_n = (a_{\lambda\mu})_{\lambda, \mu \in P_n}$ can be regarded as a common extension of the matrix Γ_n of Stembridge coefficients and the decomposition matrix $D_n^{(2)}$.

Theorem 4.6.

$$|\det A_n| = 2^{k_n},$$

where $k_n = \sum_{\lambda \in P_n} \ell(\lambda^e) = \sum_{\lambda \in P_n} (\ell(\tilde{\lambda}^r) - \ell(\lambda^r))$.

Proof. We have four bases of $\mathcal{F}(0)_n$; $S = (S_\lambda(x))_{\lambda \in P_n}$, $\tilde{S} = (S_\lambda(x, x))_{\lambda \in P_n}$, $V = (P_{\lambda^r}(x)S_{\lambda^d}(x^2))_{\lambda \in P_n}$ and $W = (Q_{\lambda^r}(x)S_{\lambda^d}(x^2))_{\lambda \in P_n}$. From Corollary 4.2, W and V are dual to each other with respect to the inner product $\langle \cdot, \cdot \rangle_{-1}$. Likewise, \tilde{S} and S are dual to each other. Hence we obtain

$${}^t M(S, V)_n M(\tilde{S}, W)_n = I,$$

where $M(S, V)_n$ denotes the transition matrix from the basis S to the basis V for $\mathcal{F}(0)_n$. Since

$$M(S, V)_n = M(S, \tilde{S})_n A_n M(W, V)_n,$$

we see that

$$(\det A_n)^2 = \frac{1}{\det M(S, \tilde{S})_n \det M(W, V)_n}.$$

Let $X_n = (\chi_\rho^\lambda)_{\lambda\rho}$ be the character table of \mathfrak{S}_n . We put $R_n = \text{diag}(z_\rho; \rho \in P_n)$ and $L_n = \text{diag}(2^{\ell(\rho)}; \rho \in P_n)$. Then we see that

$$\begin{aligned} \det M(S, \tilde{S})_n &= \det M(S, p)_n \det M(p, \tilde{S})_n \\ &= \det X_n R_n^{-1} \det L_n^{-1} {}^t X_n \\ &= \det L_n^{-1}, \end{aligned}$$

and

$$\det M(W, V)_n = \prod_{\lambda \in P_n} 2^{\ell(\lambda^r)}.$$

Hence we have

$$\det A_n^2 = \prod_{\lambda \in P_n} 2^{\ell(\lambda) - \ell(\lambda^r)} = \prod_{\lambda \in P_n} 2^{2\ell(\lambda^d)} = \prod_{\lambda \in P_n} 2^{2\ell(\lambda^e)}.$$

\square

Here is a small list of k_n .

| | | | | | | | | | |
|-------|---|---|---|---|---|----|----|----|---------|
| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | \dots |
| k_n | 0 | 1 | 1 | 4 | 5 | 11 | 15 | 28 | \dots |

Next we consider the ‘‘Cartan-like’’ matrix ${}^t A_n A_n$. The Frobenius formula for W_λ reads

$$p_\sigma p_{2\rho} = \sum_{\lambda \in P_{n_0, n_1}} 2^{-\ell(\lambda^r)} X_\sigma^{\lambda^r} \chi_\rho^{\lambda^d} W_\lambda(x)$$

for $\sigma \in OP_{n_0}$ and $\rho \in P_{n_1}$, where the Green function X_σ^λ is defined by

$$Q_\lambda(x) = \sum_{\sigma} 2^{\ell(\sigma)} z_\sigma^{-1} X_\sigma^\lambda p_\sigma$$

for $\lambda \in SP_n$. This formula shows that the transition matrix $M(p, W)_n$ is, after a suitable sorting of rows and columns, decomposed into diagonal blocks, each block indexed by the pair (n_0, n_1) with $n_0 + 2n_1 = n$.

We have

$$\begin{aligned} {}^t A_n A_n &= {}^t M(p, W)_n {}^t M(\tilde{S}, p)_n M(\tilde{S}, p)_n M(p, W)_n \\ &= {}^t M(p, W)_n ({}^t L_n X_n^{-1}) (X_n R_n^{-1} L_n) M(p, W)_n \\ &= {}^t M(p, W)_n {}^t L_n R_n^{-1} L_n M(p, W)_n. \end{aligned}$$

Since $L_n^2 R_n^{-1}$ is diagonal matrix, ${}^t A_n A_n$ is block diagonal matrix, each block indexed by the pair (n_0, n_1) . Let denote B_{n_0, n_1} the corresponding block in ${}^t A_n A_n$. Note that the “principal” block $B_{n, 0}$ is nothing but the matrix G_n .

Example 4.7.

$$\begin{aligned} {}^t A_3 A_3 &= \begin{matrix} & (3, \emptyset) & (21, \emptyset) & (1, 1) \\ \begin{matrix} (3, \emptyset) \\ (21, \emptyset) \\ (1, 1) \end{matrix} & \begin{pmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{matrix}, \\ {}^t A_4 A_4 &= \begin{matrix} & (4, \emptyset) & (31, \emptyset) & (\emptyset, 2) & (\emptyset, 1^2) & (2, 1) \\ \begin{matrix} (4, \emptyset) \\ (31, \emptyset) \\ (\emptyset, 2) \\ (\emptyset, 1^2) \\ (2, 1) \end{matrix} & \begin{pmatrix} 4 & 2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \end{matrix}. \end{aligned}$$

Theorem 4.8.

$$|\det B_{n_0, n_1}| = 2^{\sum_{\lambda \in P_{n_0, n_1}} (\ell(\tilde{\lambda}^r) + \ell(\lambda^d) - \ell(\lambda^r))}.$$

Proof. We have

$$\begin{aligned} {}^t A_n A_n &= {}^t M(p, W)_n {}^t L_n R_n^{-1} L_n M(p, W)_n \\ &= {}^t M(p, W)_n {}^t L_n R_n^{-1} L_n M(p, \tilde{p})_n M(\tilde{p}, V)_n M(V, W)_n \\ &= {}^t M(p, W)_n {}^t L_n R_n^{-1} M(\tilde{p}, V)_n M(V, W)_n, \end{aligned}$$

where $\tilde{p}_\rho(x) = p_\rho(x, x)$. Note that ${}^t L_n$, R_n^{-1} , $M(V, W)_n$ are diagonal matrices. By requiring that $\langle p_\rho, p_\sigma \rangle_{-1} = 2^{-\ell(\rho)} z_\rho \delta_{\rho\sigma}$, we obtain $\langle p_\rho, \tilde{p}_\sigma \rangle_{-1} = 2^{\ell(\sigma) - \ell(\rho)} z_\rho \delta_{\rho\sigma}$. Hence,

$$\det(M(\tilde{p}, V)_n {}^t M(p, W)_n Z_n^{-1}) = \det I.$$

We recall that ${}^t A_n A_n$ is block diagonal. We have

$$|\det B_{n_0, n_1}| = 2^{\sum_{\lambda \in P_{n_0, n_1}} (\ell(\tilde{\lambda}^r) + \ell(\lambda^d) - \ell(\lambda^r))}.$$

□

For the principal block $B_{n,0}$, we have

$$|\det B_{n,0}| = 2^{\sum_{\lambda \in SP_n} (\ell(\tilde{\lambda}) - \ell(\lambda))}.$$

We conclude this note with an inner product expression of ${}^t A_n A_n$.

Proposition 4.9.

$${}^t A_n A_n = \left(\langle P_{\lambda^r}(x), P_{\mu^r}(x) \rangle_0 \langle S_{\lambda^d}(x^2), S_{\mu^d}(x^2) \rangle_0 \right)_{\lambda, \mu}.$$

Proof. We have already given

$${}^t A_n A_n = {}^t M(p, W)_n L_n^2 R_n^{-1} M(p, W)_n.$$

Hence

$$\begin{aligned} & \sum_{\sigma, \rho} 2^{-\ell(\lambda^r) - \ell(\mu^r)} X_{\sigma}^{\lambda^r} X_{\sigma}^{\mu^r} \chi_{\rho}^{\lambda^d} \chi_{\rho}^{\mu^d} 2^{2\ell(\sigma) + 2\ell(\rho)} z_{\sigma}^{-1} z_{\rho}^{-1} \\ &= 2^{-\ell(\lambda^r) - \ell(\mu^r)} \sum_{\sigma, \rho} (2^{2\ell(\sigma)} X_{\sigma}^{\lambda^r} X_{\sigma}^{\mu^r} z_{\sigma}^{-1}) (2^{\ell(\rho)} \chi_{\rho}^{\lambda^d} \chi_{\rho}^{\mu^d} z_{\rho}^{-1}) \\ &= 2^{-\ell(\lambda^r) - \ell(\mu^r)} \langle Q_{\lambda^r}, Q_{\mu^r} \rangle_0 \langle S_{\lambda^d}(x^2), S_{\mu^d}(x^2) \rangle_0. \end{aligned}$$

□

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